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Continuum Percolation Conductivity Exponents in Restricted Domains

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Conductivity behavior of continuum percolation in restricted two-dimensional domains is simulated by considering systems of randomly distributed disks. The domain is restricted in that conducting objects are permitted to lie in only a portion of the domain. Such a restricted domain might better approximate some natural systems. Simulations of two-dimensional systems, based on three distributions of local conductances, are examined and found to demonstrate a power-law behavior with conductivity exponents smaller than those arising in regular lattice and continuum percolation.

KEY WORDS: Percolation; restricted domains; conductivity exponents; Monte Carlo; conductance distribution.

1. INTRODUCTION

In recent years, there has been extensive study of percolation in both two-dimensional and three-dimensional systems.[1] In particular, percolation theory has been used to analyze conductivity properties of disordered systems, and it has been suggested that these systems can characterize properties of many natural systems, including porous and fractured media.[2,3]

Percolation theory provides an expression relating the conductance of a system to the volumetric fraction or density of the conducting phase, in the form, for example, $K \propto (N - N_c)^t$ where K is the overall system conductivity, N is the number of conducting objects in the domain, N_c is the

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critical number of such objects for the onset of percolation, and t is a conductivity exponent specific to the quantity K. The universal conductivity exponents for random, isotropic systems are $t \approx 1.3$ in two-dimensional systems and $t \approx 2.0$ in three-dimensional systems.[4]

Previous studies have analyzed nonuniversal exponents in both the hydraulic and electrical conductivity of various systems.[3,5,6] In general, the focus has been on two issues—the description of local conductivities, and the conceptualization of the conducting phase geometry. It has been found, for example, that nonuniversal exponents may be expected to arise if the local conductivity distribution diverges as the local conductivities approach zero, with a critical behavior characterized by exponents that are always larger than the universal exponents.[5] Monte Carlo simulation results for both electrical and hydraulic conductivity in two- and three-dimensional systems have been shown to be in agreement with these predictions.[3,6]

The manner in which conducting phase (pore space) geometry is constructed can also lead to variations in behavior and to deviation of exponents from universal values. Broadly speaking, in terms of application to porous materials, there are two frequently applied constructions of pore space geometry: (1) "inverted random void" (or "inverted Swiss cheese") models, wherein the objects under consideration represent the pore spaces,[3,6,7] and (2) "random void" (or "Swiss cheese") models, wherein the considered objects represent the solid grains of the matrix.[3,8] These constructions lead to different analytical developments and exponent values. However, as for the case of diverging distributions of local conductivities,[5] the various nonuniversal exponents that arise in both the random void and inverted random void constructions are larger than the universal exponents.

It has also been shown that exponents either larger or smaller than the universal values—although always larger than unity—arise in anisotropic systems, depending on the direction of anisotropy in relation to the direction of measurement.[2,9,10] These results suggest that three-dimensional systems can yield two-dimensional exponents due to the geometry, not of the total system, but of the conducting system.

In all studies to date of nonuniversal behavior, the geometry of the percolating system has been assumed to be two dimensional or three dimensional, with the conducting phase randomly filling any part of the two-dimensional or three-dimensional space. There is, however, increasing reference to the fact that many natural systems cannot be adequately described as two dimensional or three dimensional, two examples being rough surfaces and fractal porous media.[11,12] This has led us to investigate percolation behavior of electrical and hydraulic conductivity in a new class of systems—restricted domains—in which conducting elements are permitted to lie in only a portion of the domain. We ask whether such restricted systems exhibit classical percolation behavior, and if so, how the critical conductivity exponents are affected.

2. THE RESTRICTED CONTINUUM MODEL

One of the usual methods[6,7] for analyzing electrical and hydraulic conductivity involves random (and unrestricted) placement of objects in a domain; a natural extension is to consider a similar percolation process in a restricted domain.

As done previously,[6,7] we consider an inverted random void model composed of conducting disks in a two-dimensional (unit square) domain. In such a model, a specified number of disks, below the percolation threshold, is first thrown into the domain. If two disks overlap, they are considered connected. The clusters that form are then considered to be the "blocking" phase (i.e., nonconducting regions). Once the prescribed number of blocking disks has been thrown, disks representing the conducting phase are thrown randomly into the domain. Again, conducting phase disks can overlap and are then considered connected, but they are not permitted to overlap with any of the disks forming the blocking phase. If a conducting phase disk happens to overlap with a blocking phase disk, it is removed from the domain. The onset of percolation then occurs at the point where there are enough disks, N_c, to form a continuous path of connected (conducting) disks from one side of the domain to the other. An example realization of a percolating cluster of conducting disks in a restricted domain is illustrated in Fig. 1.

For simplicity, disks forming both blocking and conducting phases are the same size. Realizations of blocker configurations that do not permit percolation clusters of conductors are excluded from the averaging. Obviously, other variations of this model, such as incorporation of a distribution of object sizes and/or shapes, will lead to considerable variability in the overall system properties.

In this inverted random void model, the fluid or electrical current flows through the "overlaps" of the nearest disks. The material that surrounds the disk is not permeable, and acts as an insulator to both fluid and electrical flow (Fig. 1). The local conductivity at the intersecting region of two disks (the so-called "neck") can then be correlated with the neck geometry, and is thus referred to as a continuum model. Each intersection (overlap) between two conducting disks is assigned its own (local) conductivity. Three cases are considered: (i) all intersections have the same (hydraulic or electrical) conductivity, (ii) the individual (hydraulic or

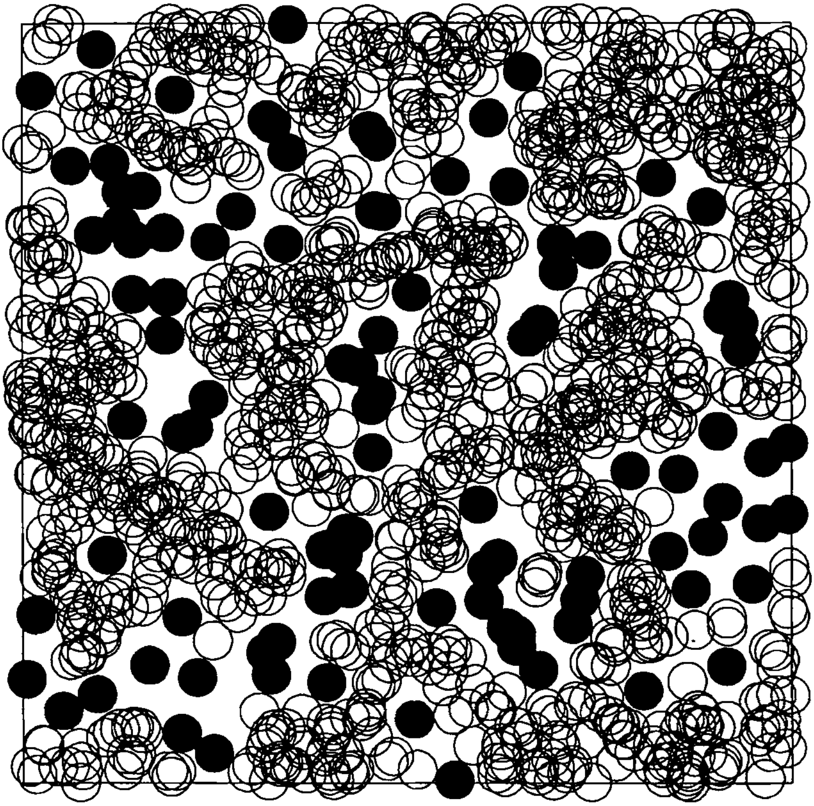

Fig. 1. Illustration of a single realization of conducting disks (open circles) embedded in the restricted domain (solid circles), exactly at threshold. In this case, the number of blocking phase disks (NBlock) is 100, and the critical number of conducting disks is 1166. Note the "channeling" effect of the blockers on the conducting phase disks.

electrical) conductivities are drawn from a log-normal distribution, and (iii) the hydraulic conductivities are determined by the degree of overlap of the disks.

For case (iii), it is clear that the value of the local conductivity is determined by the narrowest region (smallest cross section) between two disks, which is the neck. It is a good approximation to assume that the local conductivity is determined by a cylinder around the neck,[3] so that the local conductivity depends on only one variable geometrical parameter, ε, which is a measure of the degree of overlap of the disks. The Hagen–Poiseuille law for the rate of fluid flow through a capillary tube is given in two dimensions by $Q = \Delta p r^2/(8\mu L)$ where Q is the rate of fluid flow, r is the radius of the capillary tube, μ is the dynamic fluid viscosity, L is the length of the tube, and Δp is the pressure difference across the tube.[13] Then we may define the local hydraulic conductivity $k \equiv Q/\Delta p$, where $k \equiv r^2/(8\mu L) \propto \varepsilon^{1/2}$.[3] In contrast, the electrical conductivity, given by $k \propto \varepsilon^0$,[3] scales as a constant, and is given by case (i). These expressions can be used to predict whether universal or nonuniversal exponents are expected to arise.[3,6] Other details of the mathematical development and procedure can be found elsewhere.[6,7]

In our simulations, conducting disks are thrown randomly into the domain until the onset of percolation. The critical number of conducting disks N_c is recorded for that particular realization, and the overall system conductivity is determined as follows. A unit discharge across the domain is assumed in one direction, and other faces are assumed impermeable. By writing a volumetric balance equation for every node, based on the various assumed local conductivities and application of Kirchhoff's law (which requires that the algebraic sum of the fluxes at a node equals zero), we obtain a set of linear algebraic equations with a symmetric coefficient matrix. These equations can be solved under the imposed boundary conditions to yield the pressure at each intersection. From the calculated pressure gradient across the inlet and outlet faces of the domain and the prescribed unit discharge across the inlet and outlet faces, the overall conductivity of the system is determined. Additional realizations of the system are then generated for larger numbers of conducting objects N over the range $1.01N_c$ to $2.0N_c$. Finally, for each of the three cases of local conductivity distributions, the calculated system conductivity is plotted against $(N/N_c - 1)$, and the critical exponent t is determined.

To the best of our knowledge, the conductivity behavior of such a restricted system has not been analyzed previously in the context of percolation. The model raises a number of interesting possibilities, but two principal questions arise immediately: (i) does a power-law behavior still exist? and (ii) if so, what are the values of the exponents? We have completed a number of calculations for disks in two-dimensional domains, and found that a power-law behavior indeed exists, with conductivity exponents that are near unity.

3. RESULTS AND DISCUSSION

We present now the results of our Monte Carlo simulations calculating hydraulic and electrical conductivity behavior in restricted two-dimensional domains. In order to make the best analysis possible, we obtained results based on averages of blocking and conducting phase realizations with percolation thresholds as large as computational constraints would permit. In all cases, the exponents are based on averages of five combined blocking and conducting phase realizations of the system, using a least squares estimate based on all the data.

Calculations were carried out for a number of restricted domains in two dimensions. For the benchmark calculation involving only one blocker (NBlock = 1), the conventional universal exponent of $t \approx 1.3$ was recovered for the three cases of local conductivities over a range of sample sizes (N_c ranging from $\approx$22,000 to $\approx$57,000). This set of realizations essentially assumed percolation in an unrestricted domain, since the effect of the single blocker (in a sample containing thousands of objects) is negligible. Realizations with larger numbers of blocking phase disks (up to NBlock $\approx$3000) for the case $N_c \approx 57{,}000$ gave similar results for the exponent. These results are consistent with conventional lattice and continuum percolation theory, in that a limited number of blocking objects would not be expected to affect seriously the distribution of the conducting phase disks.

As the number of blocking disks is further increased, however, a drop in the conductivity exponents is found. For the largest sample sizes we considered ($N_c \approx 57{,}000$), the maximum number of blocking disks (over a

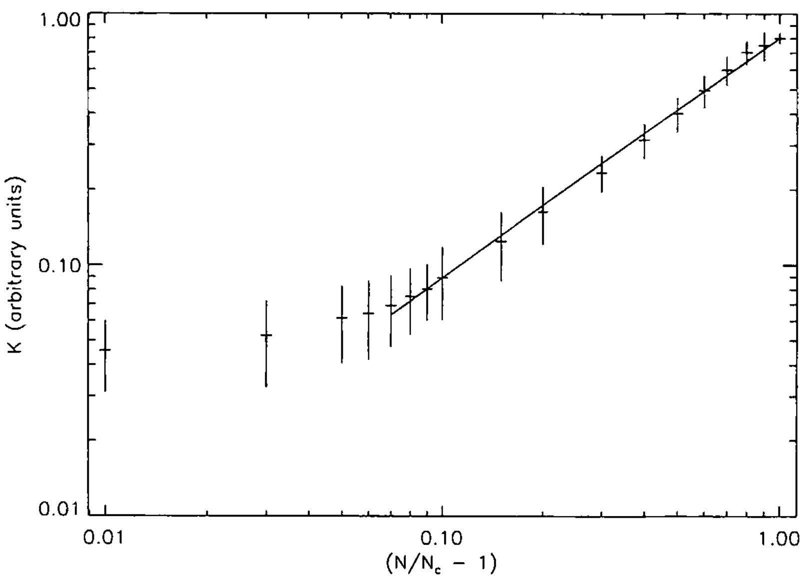

Fig. 2. Dependence of sample conductivity on proximity to the percolation threshold in the two-dimensional system, for $N_c \approx 57,000$ and NBlock $\approx$6000. For clarity, data are shown (average values and error bars of one standard deviation) for the case of constant local conductivity. The slope of the regression line is 0.99.

number of realizations) that will still permit generation of a percolating cluster of the conducting phase is NBlock ≈ 6000. The calculated exponents for the three cases of local conductivity are all $t = 0.99 \pm 0.12$ (see Fig. 2).

As might be expected, changing the effective dimension of the connected, conducting phase changes the associated critical conductivity exponent. To demonstrate that the exponents found here are not an artifact of the finite sample size, similar calculations were made for smaller sample sizes. For samples with $N_c \approx 22{,}000$ and $N_c \approx 38{,}000$, the same exponents ($t \approx 0.99$) are calculated. As N_c decreases, however, fewer points lie on the regression line with slope 0.99; for $N_c \approx 22{,}000$, points lie on the line only for $N \geqslant 1.2N_c$ (in contrast to points much closer to N_c as shown in Fig. 2). Moreover, the confidence bounds (standard error estimate of the regression) decreased (from ± 0.17 for the runs with $N_c \approx 22{,}000$, to ± 0.12) as the sample size increased, due to decreased variability in the computed overall conductivity over the realizations. Similar arguments have been used previously to examine the influence of finite sample sizes on calculated exponents.[1,6]

It is well known that the critical percolation area fraction for randomly placed disks is $\phi_c \approx 0.68$.[2] In our case, for the largest sample sizes ($N_c \approx 57{,}000$) with NBlock ≈ 6000, the blocking phase area fraction is $\phi \approx 0.12$. However while the blocking phase is far from the critical area fraction (and thus well below the number of blocking disks required to prevent occurrence of a percolating cluster of the conducting phase), increasing NBlock further made infeasible the computational effort required (over a number of realizations) to permit percolating clusters of the conducting phase disks to form.

To illustrate the influence of the blockers, even well below the critical area fraction for the blocking phase, Figs. 1 and 3 show realizations of embedded conducting disks in restricted and unrestricted domains, where the number of blocking disks is 100 and 0, respectively. Figure 3 represents a set of percolation clusters obtained in the usual unrestricted domain. The reduced density of disks, in comparison to Fig. 1, is readily apparent. The effect of the blocking phase is to channel the conducting disks, thus increasing the overall density of the objects. The lower conductivity exponent indicates a reduced rate of increase in electrical and hydraulic conductivity as additional conducting disks are added to the system. As the number of conducting disks increases, the number of overlaps increases, but no new conducting paths are established, i.e., additional conducting finite clusters are not joining the percolating cluster, so that the overall conductivity does not increase significantly. This means that only a few locations effectively control the flow.

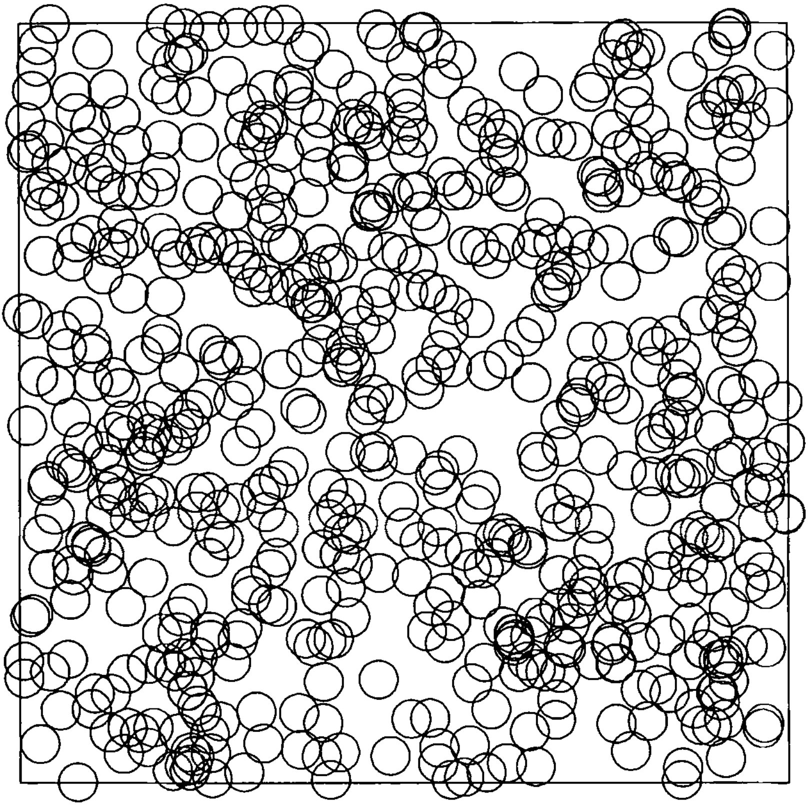

Fig. 3. Illustration of a single realization of conducting disks (open circles) embedded in an unrestricted domain, exactly at threshold. In this case, the critical number of conducting disks is 637. Note the reduced density of the disks in comparison to Fig. 1.

We also find that N_c increases with the number of blockers, because only a very limited domain is open to the conducting disks, and the probability of throwing randomly a small number of disks to form a percolating cluster is low, i.e., a much larger number of disks must be thrown in order to "cover" the reduced open domain. For example, as shown in Fig. 1, $N_c = 1166$ for a system with NBlock = 100; in contrast, if NBlock = 0 for a similar system, the critical density drops significantly to $N_c = 637$ (Fig. 3). Similar variability in N_c was found for all simulations discussed here.

4. CONCLUSIONS

We have considered a new type of continuum percolation in a restricted domain, and analyzed conductivity behavior by simulating systems of randomly distributed disks. We find that power-law behavior indeed arises with two-dimensional conductivity exponents approximately equal to unity. This value is lower than the universal conductivity exponent in lattice and continuum percolation systems.

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REFERENCES

1. D. Stauffer, *Introduction to Percolation Theory* (Taylor & Francis, London, 1985).
2. I. Balberg, Excluded-volume explanation of Archie's law, *Phys. Rev. B* **33**(5):3618–3620 (1988).
3. S. Feng, B. I. Halperin, and P. N. Sen, Transport properties of continuum systems near the percolation threshold, *Phys. Rev B* **35**(1):197–214 (1987).
4. A. Aharony, Percolation, in *Directions in Condensed Matter Physics*, G. Grinstein and G. Mazenko, (eds.) (World Scientific, Singapore, 1986), pp. 1–50.
5. P. M. Kogut and J. Straley, Distribution-induced non-universality of the percolation conductivity exponents, *J. Phys. C: Solid State Phys.* **12**2151–2159 (1979).
6. B. Berkowitz and I. Balberg, Percolation approach to the problem of hydraulic conductivity in porous media, *Transp. Porous Media* **9**(3):275–286 (1992).
7. I. Balberg, N. Wagner, D. W. Hearn, and J. A. Ventura, Critical behavior of the electrical resistance and its noise in inverted random-void systems, *Phys. Rev. Lett.* **60**(19):1887–1890 (1988).
8. B. I. Halperin, S. Feng, and P. N. Sen, Differences between lattice and continuum percolation transport exponents, *Phys. Rev. Lett.* **54**(22):2391–2394 (1985).
9. I. Balberg and N. Binenbaum, Cluster structure and conductivity of three-dimensional continuum systems, *Phys. Rev. A* **31**(2):1222–1225 (1985).
10. K. S. Mendelson and M. H. Cohen, The effect of grain anisotropy on the electrical properties of sedimentary rocks, *Geophysics* **47**(2):257–263 (1982).
11. A. J. Katz and A. H. Thompson, Fractal sandstone pores: Implications for conductivity and pore formation, *Phys. Rev. Lett.* **54**(12):1325–1328 (1985).
12. A. H. Thompson, A. J. Katz, and C. E. Krohn, The microgeometry and transport properties of sedimentary rock, *Adv. Phys.* **36**(5):625–694 (1987).
13. G. K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge University Press, Cambridge, 1967).